

CHARACTERISTIC POLYHEDRA OF SINGULARITIES WITHOUT COMPLETION - PART II

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ABSTRACT. Let $J \subset R$ be a non-zero ideal in a regular local Noetherian ring R and $(u) = (u_1, \dots, u_e)$ system of regular elements in R which can be extended to a regular system of parameters for R . In [H] Hironaka associates to this data a polyhedron $\Delta(J; u)$ reflecting the nature of the singularity given by J , the so called characteristic polyhedron of J with respect to (u) . Moreover, he proved that one compute $\Delta(J; u)$ in certain good situations by passing to the completion.

In this article we prove that Hironaka's characteristic polyhedron can be achieved without passing to the completion if R is excellent and the ideal of the reduced ridge of J' coincides with the ideal of the directrix of J' , where $J' = J \cdot R'$ and $R' = R/\langle u \rangle$.

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INTRODUCTION

Let $(R, M, K = R/M)$ be a Noetherian regular local ring, $J \subset R$ a non-zero ideal and $(u) = (u_1, \dots, u_e)$ a system of regular elements in R . In [H] Hironaka associates a polyhedron $\Delta(J; u)$ to this situation, the so called characteristic polyhedron of $(J; u)$, which is an important tool for the study of singularities. For example, it appears in his proof for resolution of singularities of excellent hypersurfaces of dimension two and also in the generalization to the case of arbitrary two dimensional excellent schemes by Jannsen, Saito and the first author [CJS]. Moreover, in [Sc3] the second author showed that the invariant introduced by Bierstone and Milman in order to give a proof for constructive resolution of singularities in characteristic zero can be purely determined by considering certain polyhedra, which are closely connected to Hironaka's polyhedron and its projections, see also [Sc1]. Further in recent work by Piltant and the first author [CP2] on the resolution of singularities of arithmetic threefolds the characteristic polyhedron plays a crucial role.

Set $R' = R/\langle u \rangle$, $M' = M \cdot R'$ and $J' = J \cdot R'$. Consider $g \in J$. Denote by $n = n_{(u)}(g) = n_{(u)}(\bar{g})$ the order of $\bar{g} = g \bmod \langle u \rangle$ in the ideal $\langle \bar{y}_1, \dots, \bar{y}_r \rangle$, where $\bar{y}_j = y_j \bmod \langle u \rangle$ for $1 \leq j \leq r$. A system of regular elements $(y) = (y_1, \dots, y_r)$ in R yields the directrix of J' if the generators of

$$I' := \text{In}_{M'}(J') := \langle \text{in}_{M'}(\bar{g}) = \bar{g} \bmod M'^{n+1} \mid \bar{g} \in J', n = n_{(u)}(\bar{g}) \rangle$$

are contained in $K[\bar{Y}], \bar{Y}_j = \bar{y}_j \bmod M'^2$, and if additionally r is minimal with this property. The ideal $\text{IDir}(J') = \langle \bar{Y} \rangle$ will be called the ideal of the directrix.

Let $(f) = (f_1, \dots, f_m)$ be a set of generators for J and $(y) = (y_1, \dots, y_r)$ a system of regular elements extending (u) to a regular system of parameters (short r.s.p.) of R . Suppose we can pick (f) such that $f_i \notin \langle u \rangle$ for all i and that the system (y) yields the directrix of J' .

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Then Hironaka proved that in the completion \widehat{R} there exist generators $(\widehat{f}) = (\widehat{f}_1, \dots, \widehat{f}_m)$ of $\widehat{J} = J \cdot \widehat{R}$ and elements $(\widehat{y}) = (\widehat{y}_1, \dots, \widehat{y}_r)$ extending (u) to a regular system of parameters for \widehat{R} such that the associated polyhedron coincides with the characteristic polyhedron,

$$\Delta(\widehat{f}; u; \widehat{y}) = \Delta(J; u).$$

In [CP1] Piltant and the first author showed that under special assumptions (R a G -ring, $m = 1$ and $r = 1$) one can attain the polyhedra without passing to the completion. Therefore it is natural to ask if this is also true in a more general situation. The main result of this article is the affirmative answer in the case of excellent rings and if additionally the reduced ridge of $J' = J \cdot R'$, $R' = R/\langle u \rangle$, coincides with its directrix.

The ridge (or *faîte* in the original French literature) is a generalization of the directrix. This is the smallest system of additive polynomials $(\sigma) = (\sigma_1, \dots, \sigma_d)$ in $\text{gr}_{M'}(R') \cong K[\overline{Y}]$ such that the generators of I' above are contained in $K[\sigma]$. Note that d is minimal with this property. The ideal $\text{IRid}(J') = \langle \sigma \rangle$ will be called the ideal of the ridge. For more details on the ridge see [G] or [BHM].

Theorem A. *Let R be an excellent Noetherian regular local ring, $J \subset R$ a non-zero ideal and $(u) = (u_1, \dots, u_e)$ a system of regular elements in R . Set $R' = R/\langle u \rangle$ and $J' = J \cdot R'$. Assume that there exists a system of regular elements (y) extending (u) to a r.s.p. for R such that the directrix of J' is determined by (y) . Moreover, suppose that the radical of the ideal of the ridge of J' coincides with the ideal of the directrix,*

$$(0.1) \quad \sqrt{\text{IRid}(J')} = \text{IDir}(J').$$

Then there exist $(z) = (z_1, \dots, z_r)$ and $(g) = (g_1, \dots, g_m)$ in R such that (u, z) is a r.s.p. for R , the system (z) yields the directrix of J' , (g) is a vertex-normalized (u) -standard basis of J , and

$$\Delta(g; u; z) = \Delta(J; u).$$

Note that condition (0.1) does always hold if we are in the situation over a perfect field. Further we want to point out that the above conditions are on the directrix of J' and not on the directrix of J . Thus the systems (y) resp. (z) are not necessarily connected to the lowest order terms of generators for J , e.g. $J = \langle y^3 + u_1^2 + u_2^7 \rangle$. This might be useful for applications apart from the resolution of singularities.

Originally, Hironaka achieves the elements (\widehat{f}) and (\widehat{y}) with $\Delta(\widehat{f}; u; \widehat{y}) = \Delta(J; u)$ by the process of vertex preparation. This procedure consists of two parts which are applied alternately: normalization of the given generators and solving vertices of the associated polyhedron. As the first one concerns certain good choices of the generators, the latter are translations of the system (y) . We split the proof of Theorem A also in two parts.

Given a (u) -standard basis (f) of J , which is a special kind of generators of J , we first show that in finitely many steps we obtain a vertex-normalized (u) -standard basis (g) of J contained in R such that $\Delta(g; u; y) = \Delta(J; u; y)$.

In general, solving the vertices of $\Delta(J; u; y)$ is not finite, see Example 1.15. Thus to finish Hironaka's procedure one has to pass to the completion. Following [CP1] we change the strategy and solve not only single vertices but whole faces of maximal dimension $e - 1$. If $\Delta(J; u) \neq \emptyset$ then there exists a measure $\Lambda(J, u, y)$ (which was already introduced in [CP1]) reflecting the difference between $\Delta(J; u; y)$ and $\Delta(J; u)$, and which strictly improves during the preparation procedure. If a face of dimension $e - 1$ is solvable then the characteristic polyhedron of the terms contributing to the face is empty. Therefore $\Delta(J; u) = \emptyset$ turns out to be the crucial case.

In fact, the assumption (0.1) is neither needed in the proof that the normalization process is finite nor in the reduction to the case of an empty characteristic polyhedron. Thus it is only required for the proof of Theorem A if $\Delta(J; u) = \emptyset$.

As a corollary Theorem A we deduce the same result for the characteristic polyhedron of an idealistic exponent introduced by the second author in [Sc2], see also [Sc1].

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1. HIRONAKA'S CHARACTERISTIC POLYHEDRA

To begin with, let us briefly recall the definition of Hironaka's characteristic polyhedron. More detailed references are section 7 of [CJS], section 2.2 of [Sc1], or Hironaka's original work [H].

Let $(R, M, K = R/M)$ be a regular local Noetherian excellent ring, $J \subset R$ a non-zero ideal and $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$ a r.s.p. of R . Set $R' = R/\langle u \rangle$, $M' = M \cdot R'$ and $J' = J \cdot R'$.

For the definition the partition of the r.s.p. is arbitrary. But the interesting point later will be when the system (y) is chosen such that they yield the directrix of J'

Definition 1.1. (1) A F -subset of $\mathbb{R}_{\geq 0}^e$ is a closed convex subset $\Delta \subset \mathbb{R}_{\geq 0}^e$ such that $v \in \Delta$ implies $v + w \in \Delta$ for every $w \in \mathbb{R}_{\geq 0}^e$.
 (2) Let $L : \mathbb{R}^e \rightarrow \mathbb{R}$ be a rational semi-positive linear form on \mathbb{R}^e . This means there are $a_i \in \mathbb{Q}_{\geq 0}$ such that $L(v_1, \dots, v_e) = \sum_{i=1}^e a_i v_i$ for $v = (v_1, \dots, v_e) \in \mathbb{R}^e$. If all a_i are positive, then L is called a positive linear form. We set

$$\Delta(L) := \{v \in \mathbb{R}^e \mid L(v) \geq 1\}.$$

The set of semi-positive resp. positive linear forms on \mathbb{R}^e will be denoted by $\mathbb{L}_0 = \mathbb{L}_0(\mathbb{R}^e)$ resp. $\mathbb{L}_+ = \mathbb{L}_+(\mathbb{R}^e)$.

(3) A F -subset $\Delta \subset \mathbb{R}_{\geq 0}^e$ is called a *rational polyhedron* if there exist finitely many rational semi-positive linear forms $L_1, \dots, L_t \in \mathbb{L}_0(\mathbb{R}^e)$ such that

$$\Delta = \bigcap_{i=1}^t \Delta(L_i).$$

(4) A point $v \in \mathbb{R}_{\geq 0}^e$ is called a *vertex* of a F -subset Δ if there exists a positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ such that

$$\{w \in \mathbb{R}^e \mid L(w) = 1\} = \{v\}.$$

We denote the set of vertices by $\text{Vert}(\Delta)$.

Definition 1.2. (1) Let $g \in R$ be an element in R , $g \notin \langle u \rangle$. Then we can expand g in a *finite* sum

$$(1.1) \quad g = \sum_{(A,B) \in \mathbb{R}_{\geq 0}^{e+r}} C_{A,B} u^A y^B$$

with coefficients $C_{A,B} \in R^\times \cup \{0\}$. Denote by $n = n_{(u)}(g)$ the order of $\bar{g} = g \bmod \langle u \rangle$ in the ideal generated by $\bar{y}_j = y_j \bmod \langle u \rangle$, $j \in \{1, \dots, r\}$. The *polyhedron associated to (g, u, y)* , denoted by $\Delta(g; u; y)$, is defined to be smallest F -subset containing all the points of the set

$$\left\{ \frac{A}{n - |B|} \mid C_{A,B} \neq 0 \wedge |B| < n \right\}.$$

(2) Let $(f) = (f_1, \dots, f_m)$ be a system of elements in R with $f_i \notin \langle u \rangle$ for every i . Then the *polyhedron associated to (f, u, y)* , denoted by $\Delta(f; u; y)$, is defined to be the smallest F -subset containing $\bigcup_{i=1}^m \Delta(f_i; u; y)$

In general, there are maybe choices for (1.1). But as it is explained in [H] at the beginning of §2 there exists a unique finite base $(A_1, B_1), \dots, (A_s, B_s)$ for the E -subset $E = \bigcup_{(A,B)} \{(A, B) + \mathbb{Z}^{e+r} \mid C_{A,B} \neq 0\}$ and this yields a unique expansion (1.1). All other monomials which might appear can be shifted into the units C_{A_i, B_i} . *Whenever we consider expansions of the form (1.1) we implicitly assume that it is of this special form.*

If we consider an ideal $J \subset R$ and generators (f_1, \dots, f_m) , then the polyhedron $\Delta(f; u; y)$ clearly depends on the choice of the generators. Let us illustrate this in the following

Example 1.3. Let $R = k[u_1, u_2, y_1, y_2]_{\langle u, y \rangle}$ for any field k . Consider the ideal $J = \langle f \rangle \subset R$ where

$$(f) = (f_1, f_2) = (y_1^2 + u_1^3, y_2^3 + u_2^7).$$

Clearly, the systems

$$(g) = (g_1, g_2) = (f_1, f_2 + f_1) = (y_1^2 + u_1^3, y_2^3 + u_2^7 + y_1^2 + u_1^3)$$

and

$$(h) = (h_1, h_2) = (f_1, f_2 + u_2^2 f_1) = (y_1^2 + u_1^3, y_2^3 + u_2^7 + u_2^2(y_1^2 + u_1^3))$$

both generate J . Then we have $\text{Vert}(\Delta(f; u; y)) = \{(\frac{3}{2}, 0); (0, \frac{7}{3})\}$ and $\text{Vert}(\Delta(g; u; y)) = \{(\frac{3}{2}, 0); (0, \frac{7}{3})\}$ and $\text{Vert}(\Delta(h; u; y)) = \{(\frac{3}{2}, 0); (0, 2); (1, \frac{2}{3})\}$. Therefore $\Delta(g; u; y) \subsetneq \Delta(f; u; y) \subsetneq \Delta(h; u; y)$.

In order to get hands on this dependence, we have to recall Hironaka's notion of a (u) -standard basis of an ideal J .

Definition 1.4. Let (R, M, K) a regular local Noetherian excellent ring with r.s.p. (u, y) as before. Consider $g = \sum C_{A,B} u^A y^B \in R$ with an expansion as in (1.1).

- (1) ([CJS] Setup A) The 0-initial form of g is defined as

$$\text{in}_0(g) := \text{in}_0(g)_{(u,y)} = \sum_{\substack{B \in \mathbb{Z}_{\geq 0}^r \\ |B|=n_{(u)}(g)}} \overline{C_{0,B}} Y^B \in K[Y],$$

where $\overline{C_{0,B}} = C_{0,B} \bmod M$.

- (2) ([CJS] Definition 6.2(2)) Let $L \in \mathbb{L}_+(\mathbb{R}^e)$ be a positive linear on \mathbb{R}^e and set

$$v_L(g) := v_L(g)_{(u,y)} := \min\{L(A) + |B| \mid C_{A,B} \neq 0\}.$$

Then we define

$$\text{in}_L(g) := \text{in}_L(g)_{(u,y)} := \sum_{\substack{(A,B) \in \mathbb{Z}_{\geq 0}^{e+r} \\ L(A) + |B| = v_L(g)}} \overline{C_{A,B}} U^A Y^B \in K[U, Y]$$

with $\overline{C_{A,B}} = C_{A,B} \bmod M$.

- (3) ([CJS] Definition 6.5) Let $(f) = (f_1, \dots, f_m)$ be a system of non-zero elements in R . A non-zero positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ is called *effective* for (f, u, y) if $\text{in}_L(f_i) \in K[Y]$ for all i .

Definition 1.5 ([H] Definition (2.20)). Let $J \subset R$ be a non-zero ideal and (u) a system of elements as before. A system of non-zero elements $(f) = (f_1, \dots, f_m)$ in J is called a (u) -standard basis of J , if there exists a system of elements $(y) = (y_1, \dots, y_r)$ extending (u) to a r.s.p. and a positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ such that $\text{in}_L(f_i) = \text{in}_0(f_i) \in K[Y]$ for all $i \in \{1, \dots, m\}$ and the following properties hold

- (1) $\text{In}_L(J) := \langle \text{in}_L(g) \mid g \in J \rangle = \langle \text{in}_0(f_1), \dots, \text{in}_0(f_m) \rangle \subset \text{gr}_M(R)$,
- (2) if $n_i := n_{(u)}(f_i) = \text{ord}_M(\text{in}_0(f_i))$, then $n_1 \leq n_2 \leq \dots \leq n_m$, and
- (3) for all $i \geq 1$ we have $\text{in}_0(f_i) \notin \langle \text{in}_0(f_1), \dots, \text{in}_0(f_{i-1}) \rangle$.

The pair (y, L) is called a *reference datum* of the (u) -standard basis.

Since $\text{in}_L(f_i) = \text{in}_0(f_i) \in K[Y]$ for all $i \in \{1, \dots, m\}$ we have by definition that L is effective for (f, u, y) .

Whenever we speak of a (u) -standard basis (f) and there are elements (y) fixed we implicitly assume that there exists a positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ such that (y, L) is a reference datum for (f) .

Note that in the previous example the system (g) is not a (u) -standard basis for J .

Let us recall the following important result on (u) -standard bases

Theorem 1.6 ([CJS] Theorem 6.9). *Let $(f) = (f_1, \dots, f_m)$ be a (u) -standard basis for an ideal $J \subset R$.*

Then, for any $(y) = (y_1, \dots, y_r)$ extending (u) to a r.s.p. for R and for any positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ which is effective for (f, u, y) , (y, L) is a reference datum for (f) .

Definition 1.7. Let $J \subset R$ be a non-zero ideal and $(u) = (u_1, \dots, u_e)$ a system of elements as before. Let $(y) = (y_1, \dots, y_r)$ be a system of elements extending (u) to a r.s.p. of R . We define

$$\Delta(J; u; y) = \bigcap_{(f)} \Delta(f; u; y),$$

where the intersection runs over all possible (u) -standard bases $(f) = (f_1, \dots, f_m)$ of J (in particular, there exists a positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ such that (y, L) is a reference datum for (f)) and further

$$\Delta(J; u) = \bigcap_{(y)} \Delta(J; u; y),$$

where the intersection ranges over all systems (y) extending (u) to a r.s.p. of R . The polyhedron $\Delta(J; u)$ is called the *characteristic polyhedron of J with respect to (u)* .

This is not Hironaka's original definition. But one can deduce from the following result by Hironaka that the two definitions coincide.

Theorem 1.8 ([H] Theorem (4.8)). *Let $J \subset R$ be a non-zero ideal and $(u) = (u_1, \dots, u_e)$ a system of elements as before. Set $R' = R/\langle u \rangle$ and $J' = J \cdot R'$. Let $(y) = (y_1, \dots, y_r)$ be a system of elements in R extending (u) to a r.s.p. of R and moreover assume that (y) yields the ideal generating the directrix of J' .*

Then there exists a (u) -standard basis $(\hat{f}) = (\hat{f}_1, \dots, \hat{f}_m)$ in \hat{R} and a system of elements $(\hat{y}) = (\hat{y}_1, \dots, \hat{y}_r)$ such that (u, \hat{y}) is a r.s.p. of \hat{R} , (\hat{y}) determines the directrix of J' and

$$\Delta(\hat{f}; u; \hat{y}) = \Delta(J; u).$$

In the proof one obtains (\hat{f}) and (\hat{y}) by applying the procedure of vertex preparation which consists of alternately normalizing the generators and solving the vertices of $\Delta(f; u; y)$. Let us recall these two processes.

We begin with normalization. First, we equip \mathbb{Z}^r with the total ordering \leq_{grlex} defined by the lexicographical order of the vector $(|B|, B_1, \dots, B_r)$ for $B \in \mathbb{Z}^r$ ($|B| = B_1 + \dots + B_r$). For $0 \neq g = \sum C_{A,B} u^A y^B \in R$ (finite expansion as in (1.1)) the *exponent of g* is defined by

$$\exp(g) := \min_{\leq_{grlex}} \{B \in \mathbb{Z}^r \mid C_{0,B} \neq 0\},$$

where the minimum is taken with respect to the total ordering mentioned above. The *exponent of an ideal $I \subset R$* is defined as the collection

$$\exp(I) := \{\exp(g) \mid 0 \neq g \in I\}.$$

Definition 1.9. Let $(f) = (f_1, \dots, f_m)$ be generators of an ideal $J \subset R$ and suppose $\exp(f_1) <_{grlex} \exp(f_2) <_{grlex} \dots <_{grlex} \exp(f_m)$. Let $f_i = \sum C_{A,B,i} u^A y^B$ be finite expansions as in (1.1) with $C_{A,B,i} \in R^\times \cup \{0\}$.

- (1) (f) is called *normalized* if $C_{A,B,i} = 0$ for every B with $B \in \exp(\langle f_1, \dots, f_{i-1} \rangle)$ and $|B| \leq n_i = n_{(u)}(f_i)$.
- (2) (f) is called *0-normalized* if the system of 0-initial forms $(in_0(f_1), \dots, in_0(f_m))$ is normalized in the sense of (1).
- (3) Let $v \in \text{Vert}(\Delta(f; u; y))$ be a vertex of $\Delta(f; u; y)$. For every $i \in \{1, \dots, m\}$ the *v-initial form* of f_i is defined as

$$in_v(f_i) := in_v(f_i)_{(u,y)} := in_0(f_i) + \sum \overline{C_{A,B,i}} U^A Y^B \in K[U, Y],$$

where the sum ranges over those $(A, B) \in \mathbb{Z}^{e+r}$ with $\frac{A}{n_{(u)}(f_i) - |B|} = v$, $\overline{C_{A,B,i}} = C_{A,B,i} \bmod M$ and $in_0(f_i)$ is the 0-initial form of f_i (Definition 1.4).

We say (f) is *normalized at the vertex v* if $(in_v(f_1), \dots, in_v(f_m))$ is normalized in the sense of (1). If (f) is normalized at every vertex of $\Delta(f; u; y)$ then we call (f) *vertex-normalized* (with respect to (u, y)).

We pointed it out only in the very last part of the definition, but all these notion depends on the choice of the r.s.p. (u, y) .

Note we have the implications: (f) is normalized $\Rightarrow (f)$ is vertex-normalized $\Rightarrow (f)$ is normalized at $v \in \text{Vert}(\Delta(f; u; y)) \Rightarrow (f)$ is 0-normalized.

Lemma 1.10. *Let $(f) = (f_1, \dots, f_m)$ and $(g) = (g_1, \dots, g_l)$ be two 0-normalized (u) -standard bases for an ideal $J \subset R$.*

Then $l = m$ and $\exp(f_i) = \exp(g_i)$ for every $i \in \{1, \dots, m\}$. In particular, we have $n_{(u)}(f_i) = n_{(u)}(g_i)$, for $1 \leq i \leq m$.

Proof. Let $L_f \in \mathbb{L}_+$, $L_f(v) = \sum_{i=1}^e a_i v_i$ with $a_i \in \mathbb{Q}_+$, be the positive linear form such that (y, L_f) is a reference datum for (f) and let $L_g \in \mathbb{L}_+$, $L_g(v) = \sum_{i=1}^e b_i v_i$ with $b_i \in \mathbb{Q}_+$, the one such that (y, L_g) is a reference datum for (g) , where $v = (v_1, \dots, v_e) \in \mathbb{R}^e$. In particular, L_f is effective for (f, u, y) (Definition 1.4) and L_g is effective for (g, u, y) .

Set $c_i := \max\{a_i, b_i\}$, for $1 \leq i \leq e$, and define $L \in \mathbb{L}_+$ by $L(v) = \sum_{i=1}^e c_i v_i$. Then L is effective for both (f, u, y) and (g, u, y) , and Theorem 1.6 implies that (y, L) is a reference datum for (f) as well as for (g) . Thus

$$\langle \text{in}_0(f_1), \dots, \text{in}_0(f_m) \rangle = \text{In}_L(J) = \langle \text{in}_0(g_1), \dots, \text{in}_0(g_l) \rangle$$

Suppose $\exp(f_1) \neq \exp(g_1)$; without loss of generality $\exp(f_1) <_{\text{grlex}} \exp(g_1)$. Then this contradicts $\text{in}_0(f_1) \in \langle \text{in}_0(g_1), \dots, \text{in}_0(g_l) \rangle$. Thus $\exp(f_1) = \exp(g_1)$.

Assume now $\exp(f_i) = \exp(g_i)$ for all $i < j$ and $\exp(f_j) \neq \exp(g_j)$ for some $j \geq 2$; without loss of generality $\exp(f_j) <_{\text{grlex}} \exp(g_j)$. Since we have $\text{in}_0(f_j) \in \langle \text{in}_0(g_1), \dots, \text{in}_0(g_l) \rangle$ there are $\mu_1, \dots, \mu_l \in K[U, Y] = \text{gr}_M(R)$ such that

$$\text{in}_0(f_j) = \mu_1 \cdot \text{in}_0(g_1) + \dots + \mu_l \cdot \text{in}_0(g_l).$$

Since $\exp(f_j) <_{\text{grlex}} \exp(g_j) <_{\text{grlex}} \exp(g_{j+1}) <_{\text{grlex}} \dots <_{\text{grlex}} \exp(g_l)$ there must exist $i < j$ with $\mu_i \neq 0$. This means there appear some g_i with $\exp(g_i) = \exp(f_i)$, $i < j$. But this is a contradiction to the property that $(\text{in}_0(f))$ is normalized.

Suppose the 0-normalized (u) -standard bases are of different length; without loss of generality $m > l$. Then $0 \neq \text{in}_0(f_{l+1}) \in \langle \text{in}_0(g_1), \dots, \text{in}_0(g_l) \rangle$. Since $\exp(g_i) = \exp(f_i)$, for all $1 \leq i \leq l$, we get again a contradiction to the assumption that $(\text{in}_0(f))$ is normalized. \square

Proposition 1.11 ([H] Lemma (3.15)). *Let $(f) = (f_1, \dots, f_m)$ be generators of an ideal $J \subset R$. Consider a r.s.p. (u, y) as before. Let $v \in \mathbb{R}_{\geq 0}^e$ be a vertex of $\Delta(f; u; y)$*

There exist $x_{ij} \in R$ such that if we set $g_1 = f_1$ and $g_i = f_i - \sum_{j=1}^{i-1} x_{ij} f_j$, for $i \geq 2$, then

- (1) (g_1, \dots, g_m) is normalized at v ,
- (2) $\Delta(g; u; y) \subset \Delta(f; u; y)$, and
- (3) $\text{Vert}(\Delta(f; u; y)) \setminus \{v\} \subset \text{Vert}(\Delta(g; u; y))$.

In Example 1.14 below we illustrate how normalization can eliminate vertices.

After normalizing the generators one has to see if vertices of the associated polyhedron can be eliminated by changes in the elements (y) .

Definition 1.12. Let $(f) = (f_1, \dots, f_m)$ be generators of an ideal $J \subset R$. Let (u, y) a r.s.p. of R such that (y) determines the directrix of $J' = J \cdot R'$, where $R' = R/\langle u \rangle$. A vertex $v \in \Delta(f; u; y)$ is called *solvable* if there exist $\lambda_j \in R^\times \cup \{0\}$, $j \in \{1, \dots, r\}$, such that we have for the system $(z) = (z_1, \dots, z_r)$, given by $z_j := y_j + \lambda_j u^v$,

$$v \notin \Delta(f; u; z).$$

In [H], Corollary (4.4.3) it is shown that if a vertex is solvable, then the images of $\lambda_j \in R^\times \cup \{0\}$ in the graded ring $\text{gr}_M(R)$ are unique.

From the definition we see that a vertex v can only be solvable if $v \in \mathbb{Z}_{\geq 0}^e$. Note that (z) has still the property that it yields the directrix of J' . Moreover, the other vertices of the polyhedron do not change under this translation. More precisely,

Proposition 1.13 ([H] Lemma (3.10)). *Let (f) , $J \subset R$ and (u, y) as in the previous definition. Let $v \in \Delta(f; u; y)$ be a solvable vertex and (z) the corresponding elements.*

Then we have

- (1) $\Delta(f; u; z) \subset \Delta(f; u; y)$,
- (2) $v \notin \Delta(f; u; z)$, and
- (3) $\text{Vert}(\Delta(f; u; y)) \setminus \{v\} \subset \text{Vert}(\Delta(f; u; z))$.

In order to normalize and to solve the vertices in a systematic way one has to equip \mathbb{R}^e with a total ordering. Then one picks the vertex that is minimal with respect to this ordering, normalizes, and tests if it is solvable. After that one takes the new minimal vertex that has not been considered yet.

In the procedure it is important to apply alternately normalization and vertex solving. In the latter we only take care of vertices and not points in the interior of the polyhedron. But still these points might be interesting after normalization.

Example 1.14. Consider the variety given by $f_1 = y_1^p$ and $f_2 = y_2^{p^2} + u^{A'} y_1^p + u^{p^2 A}$ over a field k of characteristic $p > 0$ and suppose $A \in \mathbb{Z}_{\geq 0}^e \cap (\frac{A'}{p^2-p} + \mathbb{Z}_{\geq 0}^e)$, $A \neq \frac{A'}{p^2-p}$. The only vertex of the associated polyhedron is given by $v := \frac{A'}{p^2-p}$ and one sees easily that v can not be solved. The normalization yields $g_2 := f_2 - u^{A'} f_1 = y_2^{p^2} + u^{p^2 A}$. Therefore the vertex v vanishes and the new vertex A is solvable via $z_2 := y_2 + u^A$.

In general, Hironaka's procedure of vertex preparation is not finite!

Example 1.15. Let k be a field of characteristic two and consider the variety given by

$$f = y^2 + y^4 + u_1^4 + u_2^7 = 0.$$

Following Hironaka's procedure we have to make the translation $y \mapsto w := y + u_1^2$ and obtain $f = w^2 + w^4 + u_1^8 + u_2^7$. Again we have to solve a vertex and clearly this is *not* a finite process. But if we consider $z := y + y^2 + u_1^2$ then we get $f = z^2 + u_2^7$ and the associated polyhedron coincides with the characteristic polyhedron.

For another example, which is valid in a more general setting, we refer the reader to Example II.5 in [CP1].

2. NORMALIZATION IS ALWAYS FINITE

Hironaka's procedure of vertex preparation splits into solving vertices and normalizing the generators of the ideal. First, we want to consider normalization.

Recall that we defined $\Delta(J; u; y) := \bigcap_{(f)} \Delta(f; u; y)$, where the intersection runs over all $(f) = (f_1, \dots, f_m)$ with the property: there is a positive linear form $L \in \mathbb{L}_+(\mathbb{R}^e)$ that makes (f) into a (u) -standard basis with reference datum (y, L) .

The aim of this section is to give a proof for

Proposition 2.1. *Let R be a regular local Noetherian excellent ring, $J \subset R$ a non-zero ideal and $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$ a r.s.p. of R such that (y) determines the directrix of $J' = J \cdot R'$, where $R' = R/\langle u \rangle$. Let $(f_1, \dots, f_m) \subset R$ be a (u) -standard basis of J with $f_i \notin \langle u \rangle$ for all $i \in \{1, \dots, m\}$.*

Then there exist $h_{ij} \in R$ such that $(g) = (g_1, \dots, g_m)$, with $g_1 = f_1$ and $g_i := f_i - \sum_{j=1}^{i-1} h_{ij} f_j$, $2 \leq i \leq m$, is a vertex-normalized (u) -standard basis of J and

$$\Delta(g; u; y) = \Delta(J; u; y).$$

We split the proof into three steps. First, we prove that the last equality holds for every vertex-normalized (u) -standard basis of the ideal J and then using this we deduce the proposition separately in the cases $\Delta(J; u; y) = \emptyset$ and $\Delta(J; u; y) \neq \emptyset$.

Lemma 2.2. *Let $(f) = (f_1, \dots, f_m)$ be a vertex-normalized (u) -standard basis of J with reference datum (y, L) for some $L \in \mathbb{L}_+$. Then*

$$\Delta(f; u; y) = \Delta(J; u; y).$$

Proof. Let (f) and (g) be two normalized (u) -standard bases. By Lemma 1.10 both have the same number of elements, $(f) = (f_1, \dots, f_m)$ and $(g) = (g_1, \dots, g_m)$, and furthermore $\exp(f_i) = \exp(g_i)$ for every $i \in \{1, \dots, m\}$.

Since (f) and (g) are generators of the same ideal, we have for every i an expression $f_i = \sum_{j=1}^m \alpha_{ij} g_j$ with certain $\alpha_{ij} \in R$. It follows from $\exp(f_1) = \exp(g_1)$ and $\exp(f_1) < \exp(g_i)$, for $i \geq 2$, that $\alpha_{11} =: \epsilon_1 \in R^\times$ is a unit. Thus $f_1 = \epsilon_1 \cdot g_1 + \sum_{j=2}^m \alpha_{1j} g_j$.

Since (f) is vertex-normalized and $\exp(f_1) = \exp(g_1)$, none of the monomials appearing in $\alpha_{21}g_1$ contributes to a vertex of $\Delta(f; u; y)$. Further (f) is 0-normalized which implies that $\alpha_{22} =: \epsilon_2 \in R^\times$ is also a unit (otherwise we do not have $\exp(f_2) = \exp(g_2)$); $f_2 = \alpha_{21}f_1 + \epsilon_2 \cdot g_2 + \sum_{j=3}^m \alpha_{ij} g_j$. By continuing we get for every i

$$(2.1) \quad f_i = \sum_{j=1}^{i-1} \alpha_{ij} g_j + \epsilon_i \cdot g_i + \sum_{j=i+1}^m \alpha_{ij} g_j.$$

with units $\epsilon_i := \alpha_{ii} \in R^\times$ and $\sum_{j=1}^{i-1} \alpha_{ij} g_j$ does not contribute to the vertices of $\Delta(f; u; y)$.

Suppose the associated polyhedra are not equal, $\Delta(f; u; y) \neq \Delta(g; u; y)$. Then there exists at least one vertex $v \in \mathbb{R}_{\geq 0}^e$ that is contained in only one of them; without loss of generality $v \in \Delta(f; u; y)$. So there is an f_i such that in its expansion, $f_i = \sum_{(A,B)} C_{A,B,i} u^A y^B$, there is (A, B) with $C_{A,B,i} \neq 0$ and $\frac{A}{n_i - |B|} = v$ ($n_i = n_{(u)}(f_i)$). Let us fix (A, B) with this property for a moment and denote them $(A(v), B(v))$.

On the other hand, we have (2.1). Hence there is α_{ij} and g_j such that the monomial $u^{A(v)} y^{B(v)}$ appears in the expansion of $\alpha_{ij} g_j$ with non-zero coefficient. This implies that the existence of $(C(v), D(v))$ such that

- (1) $(A(v), B(v)) \in (C(v), D(v)) + \mathbb{R}_{\geq 0}^{e+r}$ and
- (2) $u^{C(v)} y^{D(v)}$ appears in the expansion of g_j with non-zero coefficient.

This implies in particular $|D(v)| \leq |B(v)| < n_i$. Property (1) yields

$$(2.2) \quad v = \frac{A(v)}{n_i - |B(v)|} \in \frac{C(v)}{n_i - |D(v)|} + \mathbb{R}_{\geq 0}^e$$

and by (2) there exists a vertex $w \in \Delta(g_j; u; y)$ such that $\frac{C(v)}{n_j - |D(v)|} \in w + \mathbb{R}_{\geq 0}^e$.

Moreover, (2.1) implies $j \geq i$ because none of the points in $\Delta(f; u; y)$ coming from $\sum_{j=1}^{i-1} \alpha_{ij} g_j$ can give a vertex of $\Delta(f; u; y)$. Thus $n_j = n_{(u)}(g_j) = n_{(u)}(f_j) \geq n_{(u)}(f_i) = n_i$. But this means $\frac{n_j - |D(v)|}{n_i - |D(v)|} \geq 1$ and

$$\frac{C(v)}{n_i - |D(v)|} = \frac{n_j - |D(v)|}{n_i - |D(v)|} \cdot \frac{C(v)}{n_j - |D(v)|} \in \frac{C(v)}{n_j - |D(v)|} + \mathbb{R}_{\geq 0}^e.$$

Together with (2.2) and the choice of w this implies $v \in w + \mathbb{R}_{\geq 0}^e \subset \Delta(g_j; u; y) \subset \Delta(g; u; y)$. This is a contradiction to the assumption $v \notin \Delta(g; u; y)$ and we get

$$(2.3) \quad \Delta(f; u; y) = \Delta(g; u; y).$$

Therefore the polyhedra of two vertex-normalized (u) -standard bases coincide.

Let now $(h) = (h_1, \dots, h_m)$ be any (u) -standard basis. By Hironaka's procedure of normalization (Proposition 1.11) we obtain a vertex-normalized (u) -standard basis (f) (possibly in \hat{R}) and $\Delta(f; u; y) \subseteq \Delta(h; u; y)$. Combining this with (2.3) completes the proof. \square

Remark 2.3. The previous lemma shows that for an arbitrary (u) -standard basis (f) of J the difference between $\Delta(f; u; y)$ and $\Delta(J; u; y)$ reflects how far (f) is away from being vertex-normalized.

Lemma 2.4. Let $(f_1, \dots, f_m) \subset R$ be a (u) -standard basis of J with $f_i \notin \langle u \rangle$ for every $i \in \{1, \dots, m\}$. Assume $\Delta(J; u; y) = \emptyset$.

Then there exist $h_{ij} \in R$ such that $(g) = (g_1, \dots, g_m)$, with $g_i := f_i - \sum_{j=1}^{i-1} h_{ij} f_j$, is a normalized (u) -standard basis of J .

Proof. Since (f) is a (u) -standard basis, we have $n_1 \leq n_2 \leq \dots \leq n_m$, where $n_i := n_{(u)}(f_i)$. By Hironaka's normalization process there are $(\hat{g}) = (\hat{g}_1, \dots, \hat{g}_m)$ in \hat{R} which build a normalized (u) -standard basis of J . Since $\Delta(J; u; y) = \emptyset$ there are no vertices and (\hat{g}) being vertex-normalized means it is normalized. By Lemma 2.2 we have $\Delta(\hat{g}; u; y) = \Delta(J; u; y) = \emptyset$. By Proposition 1.11 $\hat{g}_1 = f_1 \in R$ and since $\Delta(\hat{g}; u; y) = \emptyset$, we must have $f_1 \in \langle y \rangle^{n_1}$. Thus $g_1 = f_1$.

In the next step we have to show that the normalization of (g_1, f_2) can be achieved in R . Set $E := \exp(g_1) \in \mathbb{Z}_{\geq 0}^r$. Then we have a finite expansions of g_1 in R

$$g_1 = D_1 \cdot y^E + \sum_{(A,B)} C_{1,A,B} u^A y^B$$

with $D_1 = C_{1,0,E} \in R^\times$ a unit and $C_{1,A,B} \in R^\times \cup \{0\}$. Further we expand $f_2 = \sum_{(A,B)} C_{2,A,B} u^A y^B$ also in a finite sum in R , $C_{2,A,B} \in R^\times \cup \{0\}$. If (g_1, f_2) is already normalized, then we set $g_2 = f_2$ and continue with (g_1, g_2, f_3) .

Suppose (g_1, f_2) is not normalized. Then there is $(A, B) \in \mathbb{Z}_{\geq 0}^{e+r}$ such that the coefficient $C_{2,A,B} \neq 0$ is a unit and $B \in E + \mathbb{Z}_{\geq 0}^r$ (i.e. $B \in \exp(\langle g_1 \rangle)$). Thus $B = E + C$ for some $C \in \mathbb{Z}_{\geq 0}^r$ and

$$(2.4) \quad \begin{aligned} u^A y^B &= u^A y^C (y^E) = u^A y^C D_1^{-1} (g_1 - \sum_{(A,B)} C_{1,A,B} u^A y^B) = \\ &= u^A y^C D_1^{-1} g_1 - \sum_{(A,B)} D_1^{-1} C_{1,A,B} u^{A+A} y^{B+C} \end{aligned}$$

If we have

$$(2.5) \quad B >_{\text{grlex}} E, \quad \text{for every } B \text{ with } C_{1,A,B} \neq 0,$$

(here \geq_{grlex} is the total order on $\mathbb{Z}_{\geq 0}^r$ given by the lexicographical order of $(|B|, B_1, \dots, B_r)$), then

$$B + C >_{\text{grlex}} E + C = B$$

and all the y -powers appearing have strictly bigger order with respect to \geq_{grlex} . Hence if we choose B minimal with respect to \geq_{grlex} , we will come after finitely many steps to the point where $|B'| > n_2$ and we stop. In particular, we have for the obtained element g_2 that $g_2 \in \langle y \rangle^{n_2}$, (g_1, g_2) is normalized and $\Delta(g_1, g_2; u; y) = \emptyset$.

In fact, (2.5) is always true: Suppose there exists B_0 such that $B_0 <_{\text{grlex}} E$ and suppose it is minimal with respect to \geq_{grlex} . (The case $B_0 =_{\text{grlex}} E$ is not interesting, because then $B_0 = E$ and we can shift $C_{A_0, B_0} u^{A_0} y^{B_0}$ into the unit D_1). Then $A_0 \neq 0$ has to be non-trivial, because otherwise $\exp(g_1) = B_0 \neq E$. Again we choose B minimal. As $g_1 \in \langle y \rangle^{n_1}$, we have $|B_0| = n_1 = |E|$. Since $B_0 <_{\text{grlex}} E$, there appears a monomial with y -power

$$B_0 + C <_{\text{grlex}} E + C = B$$

after the change given by (2.4). Set $B' := B_0 + C = B - E + B_0$. If $B' \in E + \mathbb{Z}_{\geq 0}^r$, then this is clearly the minimal one and we repeat the replacement via (2.4) and obtain $B'' = B - 2(E - B_0)$. (Note that $E - B_0 >_{\text{grlex}} (0)$). After finitely many of these steps we obtain $B^* \notin E + \mathbb{Z}_{\geq 0}^r$. Then there is a monomial $C_{A^*, B^*} u^{A^*} y^{B^*}$ that gives a point v in the polyhedron $\Delta(g_1, f_2^*; u; y)$ (where f_2^* denotes the modified f_2 obtained up to this step). Moreover, v can not be eliminated by further normalization. Therefore it also appears in the polyhedron of the final normalized (u) -standard basis (\hat{g}) . But this contradicts $\Delta(\hat{g}; u; y) = \emptyset$. Hence the assumption $\Delta(J; u; y) = \emptyset$ implies that there is no (A_0, B_0) with $B_0 <_{\text{grlex}} E$ appearing in the expansion of f_1 .

By the previous argument we can normalize (g_1, f_2) in R and obtain $(g_1, g_2) \subset R$. The element h_{21} is defined via the finitely many changes of the kind (2.4).

Now we repeat the algorithm for (g_1, g_2, f_3) and so on until we reach the desired normalized (u) -standard basis (g_1, \dots, g_m) in R .

Recall that we have by Lemma 2.2 $\Delta(g; u; y) = \Delta(J; u; y) = \emptyset$. □

Unfortunately, normalization in our strong sense is not necessarily finite if the polyhedron is non-empty, $\Delta(J; u; y) \neq \emptyset$. A counterexample is given below. Therefore we follow Hironaka and stop our normalization process as soon as we reach the polyhedron $\Delta(J; u; y)$, i.e. we stop when (f) is vertex-normalized.

Example 2.5. Let k be a field of characteristic two and consider the ideal $J := \langle f_1, f_2 \rangle \subset k[u, y, z]_{\langle u, y, z \rangle}$, where

$$f_1 = y^3 + y^4 u + y^2 u^2 + u^5 \quad \text{and} \quad f_2 = z^5 + y^3 u.$$

The normalization process tells us to replace $y^3 = f_1 + y^4 u + y^2 u^2 + u^5$ in f_2 and we obtain

$$g_2 = z^5 + y^4 u^2 + y^2 u^3 + u^6.$$

The monomial $y^2 u^3$ yields the vertex $1 \in \Delta(f_1, g_2; u; y, z)$ which does not vanish by further normalization and is even not solvable. Thus we have $\Delta(f_1, g_2; u; y, z) = \Delta(J; u)$.

But (f_1, g_2) is not normalized in our stronger sense. Again we would have to replace $y^3 = f_1 + y^4 u + y^2 u^2 + u^5$ and get

$$h_2 = z^5 + y^5 u^3 + y^3 u^4 + y u^7 + y^2 u^3 + u^6$$

Again there appears y^3 and we run into a loop. But the polyhedron does not change any more!

Lemma 2.6. *Let $(f_1, \dots, f_m) \subset R$ be a (u) -standard basis of J with $f_i \notin \langle u \rangle$ for all $i \in \{1, \dots, m\}$. Assume $\Delta(J; u; y) \neq \emptyset$.*

Then there exist $h_{ij} \in R$ such that $(g) = (g_1, \dots, g_m)$, with $g_i := f_i - \sum_{j=1}^{i-1} h_{ij} f_j$, is a vertex-normalized (u) -standard basis of J .

Proof. We begin as in the proof of the empty case. By Hironaka's result and Lemma 2.2 there is a vertex-normalized (u) -standard basis of J in \hat{R} , denoted $(\hat{g}) = (\hat{g}_1, \dots, \hat{g}_m)$, such that $\Delta(\hat{g}; u; y) = \Delta(J; u; y)$. Moreover, $g_1 := \hat{g}_1 = f_1 \in R$.

Next we normalize (g_1, f_2) in R . Set $E := \exp(g_1) \in \mathbb{Z}_{\geq 0}^r$ and consider again a finite expansions of g_1 in R ,

$$g_1 = D_1 \cdot y^E + \sum_{(A,B)} C_{1,A,B} u^A y^B$$

with $D_1 = C_{1,0,E} \in R^\times$ a unit and $C_{1,A,B} \in R^\times \cup \{0\}$. Further we also expand $f_2 = \sum_{(A,B)} C_{2,A,B} u^A y^B$ in a finite sum in R , $C_{2,A,B} \in R^\times \cup \{0\}$.

Suppose (g_1, f_2) is not normalized. Then there is $(\mathcal{A}, \mathcal{B}) \in \mathbb{Z}_{\geq 0}^{e+r}$ such that the coefficient $C_{2,\mathcal{A},\mathcal{B}} \neq 0$ is a unit and $\mathcal{B} \in E + \mathbb{Z}_{\geq 0}^r$ (i.e. $\mathcal{B} \in \exp(\langle g_1 \rangle)$). Thus $\mathcal{B} = E + C$ for some $C \in \mathbb{Z}_{\geq 0}^r$ and by (2.4)

$$(2.6) \quad u^{\mathcal{A}} y^{\mathcal{B}} = u^{\mathcal{A}} y^C D_1^{-1} g_1 - \sum_{(A,B)} D_1^{-1} C_{1,A,B} u^{A+\mathcal{A}} y^{B+C}$$

Now we measure the difference between $\Delta(g_1, f_2; u; y)$ and $\Delta(J; u; y)$ in the following way. Let $L : \mathbb{R}^e \rightarrow \mathbb{R}$, $L(v_1, \dots, v_e) = \sum_{i=1}^e a_i v_i$ for $a_i \in \mathbb{Q}_{\geq 0}$, be one of the finitely many semi-positive linear forms defining the faces of $\Delta(J; u; y)$. We set

$$\delta_L := \min\{L(v) \mid v \in \Delta(g_1, f_2; u; y)\} \leq 1.$$

If $\delta_L = 1$, then we are already at $\Delta(J; u; y)$ and we consider the next linear form. Therefore suppose $\delta_L < 1 \in \frac{1}{n_2! \alpha!} \mathbb{Z}_{\geq 0}$, where $\alpha \in \mathbb{Z}_{\geq 0}$ is the biggest denominator in the coefficients defining L . We rewrite the finite expansion of f_2 as follows:

$$(2.7) \quad f_2 = \sum_{|B| \geq n_2} C_{2,A,B} u^A y^B + \sum_{\frac{L(A)}{n_2 - |B|} = \delta_L} C_{2,A,B} u^A y^B + \sum_{\frac{L(A)}{n_2 - |B|} > \delta_L} C_{2,A,B} u^A y^B,$$

where we abbreviate $\sum_{(*)}$ for the sum ranging over those $(A, B) \in \mathbb{Z}_{\geq 0}^{e+r}$ such that the condition $(*)$ holds.

We choose \mathcal{B} above minimal with respect to \geq_{grlex} and such that additionally $C_{2,\mathcal{A},\mathcal{B}} \neq 0$ in the sum in the middle of (2.7). By (2.6) $v := \frac{\mathcal{A}}{n_2 - |\mathcal{B}|}$ yields at most the points

$$w := \frac{\mathcal{A} + A}{n_2 - |C| - |B|} = \frac{n_2 - |\mathcal{B}|}{n_2 - |C| - |B|} \cdot \frac{\mathcal{A}}{n_2 - |\mathcal{B}|} + \frac{A}{n_2 - |C| - |B|}$$

with (A, B) coming from f_1 . If $|C| + |B| \geq n_2$, then this point does not appear in $\Delta(f_2; u; y)$. So we may assume $|C| + |B| < n_2$. Set

$$\rho := \frac{n_2 - |\mathcal{B}|}{n_2 - |C| - |B|} = \frac{n_2 - |C| - |E|}{n_2 - |C| - |B|} = 1 + \frac{|B| - |E|}{n_2 - |C| - |B|}.$$

Then we have the following cases:

(1) If $|B| > |E|$, then $\rho > 1$ and

$$L(w) = \rho \cdot L(v) + \frac{L(A)}{n_2 - |C| - |B|} \geq \rho \cdot L(v) > L(v) = \delta_L$$

(2) If $|B| < |E|$, then $\rho < 1$. Recall that $|E| = n_1$ and hence $|B| < n_1$. Therefore the point $\frac{A}{n_1 - |B|} \in \Delta(g_1; u; y) \subset \Delta(J; u; y)$ appears in the polyhedron associated to J . By the choice of L as linear form defining $\Delta(J; u; y)$ we have $\frac{L(A)}{n_1 - |B|} \geq 1 > \delta_L$. Using this, the definition of ρ and $|E| = n_1$ we obtain

$$\begin{aligned} L(w) &= \rho \cdot \delta_L + \frac{n_1 - |B|}{n_2 - |C| - |B|} \cdot \frac{L(A)}{n_1 - |B|} > \\ &> \frac{n_2 - |C| - |E|}{n_2 - |C| - |B|} \cdot \delta_L + \frac{n_1 - |B|}{n_2 - |C| - |B|} \cdot \delta_L = \delta_L. \end{aligned}$$

(3) If $|B| = |E|$, then $\rho = 1$. Either $A = 0$ and thus $L(w) = L(v) = \delta_L$. But then we must have $B >_{\text{grlex}} E$ by the definition of E as the exponent of $g_1 = f_1$. Therefore $B + C >_{\text{grlex}} E + C = \mathcal{B}$.

Or $A \neq 0$. Recall that $L(v_1, \dots, v_e) = \sum_{i=1}^e a_i v_i$ for $a_i \in \frac{1}{\alpha!} \mathbb{Z}_{\geq 0} \subset \mathbb{Q}_{\geq 0}$.

(a) If $L(A) > 0$ is positive, then

$$L(w) = \rho \cdot L(v) + \frac{L(A)}{n_2 - |C| - |B|} > L(v) = \delta_L.$$

(b) Suppose $L(A) = 0$. Together with $|B + C| = |B| + |C| = |E| + |C| = |\mathcal{B}|$ this implies

$$L(w) = \frac{L(\mathcal{A} + A)}{n_2 - |B + C|} = \frac{L(\mathcal{A}) + L(A)}{n_2 - |\mathcal{B}|} = \frac{L(\mathcal{A})}{n_2 - |\mathcal{B}|} = \delta_L.$$

If $B >_{\text{grlex}} E$, then $B + C >_{\text{grlex}} E + C = \mathcal{B}$.

But it may happen that $B <_{\text{grlex}} E$. This does not contradict $E = \exp(g_1)$, because the monomial corresponding to B might be $u^A y^B$ with $A \neq 0$ and $L(A) = 0$. Thus the monomial $u^A y^B$ could give (via (2.6)) the monomial $u^{A+A} y^{C+B}$ with $|C + B| = |\mathcal{B}|$ and $C + B <_{\text{grlex}} \mathcal{B}$. We start the process again. We expand the new f'_2 as in (2.7) and pick \mathcal{B}' in the second sum minimal with respect to \geq_{grlex} . As \mathcal{B}' was minimal, we must have $\mathcal{B}' = C + B$. If $\mathcal{B}' \in E + \mathbb{Z}_{\geq 0}^r$, then use (2.6), repeat everything and obtain $(\mathcal{A}'', \mathcal{B}'')$ with $\mathcal{B}'' <_{\text{grlex}} \mathcal{B}' <_{\text{grlex}} E$ and $|\mathcal{B}''| = |\mathcal{B}'| = |E|$ and $\frac{L(\mathcal{A}'')}{n_2 - |\mathcal{B}''|} = \delta_L$. Obviously, this stops after finitely many steps; without loss of generality it stops for $(\mathcal{A}'', \mathcal{B}'')$ and denote by f''_2 the respective new form of f_2 . But then there is a point $w'' = \frac{\mathcal{A}''}{n_2 - |\mathcal{B}''|} \in \Delta(g_1, f''_2; u; y)$ with $L(w'') = \delta_L < 1$ and which does not vanish by the further normalization procedure, i.e. $w'' \in \Delta(J; u; y)$. This is a contradiction to the choice of L as a linear form defining a face of $\Delta(J; u; y)$! Therefore the case $L(A) = 0$ and $B <_{\text{grlex}} E$ can not appear.

This means the points which may appear are

- either strictly closer to $\Delta(J; u; y)$ ($L(w) > \delta_L$),
- or remain in the same difference ($L(w) = L(v) = \delta_L$) and $B + C >_{\text{grlex}} \mathcal{B}$.

Recall that we have chosen \mathcal{B} minimal with respect to \geq_{grlex} .

Now we expand the new f'_2 as in (2.7). By the previous argument the minimal $\mathcal{B}' \in E + \mathbb{Z}_{\geq 0}^r$ appearing in the sum in the middle has larger order with respect to \geq_{grlex} than \mathcal{B} , so $\mathcal{B}' >_{\text{grlex}} \mathcal{B}$. (In fact, the minimal \mathcal{B}' appearing in the sum in the middle must automatically fulfil $\mathcal{B}' \in E + \mathbb{Z}_{\geq 0}^r$, because this the monomial corresponds to a point that is not appearing in $\Delta(J; u; y)$). Since \geq_{grlex} is defined via the lexicographical order of $(|B|, B_1, \dots, B_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, we see that after finitely many steps we reach the point where $|\mathcal{B}''| > n_2$ and in particular the sum in the middle in (2.7) is empty. For the new f''_2 we have $\delta_L(\Delta(g_1, f''_2; u; y)) > \delta_L(\Delta(g_1, f_2; u; y))$. This means we are strictly closer to $\Delta(J; u; y)$.

Since $\delta_L(\cdot) \in \frac{1}{n_2! \alpha!} \mathbb{Z}_{\geq 0}$ takes only values in a discrete set, we obtain after finitely many steps f_2^* such that $\delta_L(\Delta(g_1, f_2^*; u; y)) \geq \delta_L(\Delta(J; u; y)) = 1$.

We apply the previous procedure for all faces and get after finitely many steps g_2 such that $\Delta(g_1, g_2; u; y) \subset \Delta(J; u; y)$. We repeat everything for (g_1, g_2, f_3) and so on until we get a (u) -standard basis (g_1, \dots, g_m) of J with the property $\Delta(g; u; y) = \Delta(J; u; y)$.

But (g) is not necessarily vertex-normalized yet (e.g. this might happen if we have $\delta_L = 1$ for every L at the beginning). Then we only have to normalize at the finitely many vertices of $\Delta(J; u; y) = \Delta(g; u; y)$ and we are done. By Hironaka's result (Proposition 1.11) this is obtained in R . Note that the whole process has only finitely many steps. Moreover, it yields $\Delta(g; u; y) = \Delta(J; u; y)$. \square

Lemma 2.2, Lemma 2.4 and Lemma 2.6 together imply Proposition 2.1.

3. REDUCTION TO THE CASE OF AN EMPTY CHARACTERISTIC POLYHEDRON

For the proof of Theorem A it remains to show that the procedure of solving vertices is also finite. In this section we reduce the proof of this to the case of an empty characteristic polyhedron. All we need to assume is

Hypothesis 3.1. Let S be a regular local Noetherian excellent ring, $I \subset S$ a non-zero ideal and $(t, x) = (t_1, \dots, t_d; x_1, \dots, x_s)$ a r.s.p. for S such that (x) determines the directrix of $I' = I \cdot S'$, where $S' = S/\langle u \rangle$. Suppose $\Delta(I; t) = \emptyset$.

Then there exist elements $(z) = (z_1, \dots, z_s)$ in S such that (t, z) is a r.s.p. for S , (z) yields the directrix of I' , and

$$\Delta(I; t; z) = \Delta(I; t) = \emptyset.$$

In contrast to Theorem A we did not assume the extra condition (0.1),

$$\sqrt{IRid(I')} = IDir(I').$$

Therefore the reduction to the case of an empty characteristic polyhedron also valid in the situation where we only require R to be excellent.

The aim of this section is to prove the following result which implies Theorem A in the case $\Delta(J; u) \neq \emptyset$ assuming that we know how to handle the case of an empty characteristic polyhedron (Hypothesis 3.1).

Proposition 3.2. *Suppose Hypothesis 3.1 is true.*

Let R be a regular local Noetherian excellent ring, $J \subset R$ a non-zero ideal and $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$ a r.s.p. of R such that (y) determines the directrix of J' . Assume $\Delta(J; u) \neq \emptyset$.

Then there exist a vertex-normalized (u) -standard basis $(g) = (g_1, \dots, g_m)$ of J , elements $(z) = (z_1, \dots, z_r)$ in R such that (u, z) is a r.s.p. for R , (z) yields the directrix of J' , and

$$\Delta(g; u; z) = \Delta(J; u).$$

For the proof of Proposition 3.2 we use the analogous measure for the difference between $\Delta(J; u; y)$ and $\Delta(J; u)$ as it is used in the proof of THEOREM II.3 in [CP1]. In fact, a variant of it is already hidden in the proof of Lemma 2.6, where we measured the difference of $\Delta(f; u; y)$ and $\Delta(J; u; y)$.

Definition 3.3. (1) Let $L \in \mathbb{L}_0(\mathbb{R}^e)$ be any rational semi-positive linear forms on \mathbb{R}^e . For a subset $\Delta \subset \mathbb{R}_{\geq 0}^e$ we set

$$\delta_L(\Delta) := \min\{L(v) \mid v \in \Delta\}.$$

- (2) Let $\emptyset \neq \Delta^\circ, \Delta^+ \subset \mathbb{R}_{\geq 0}^e$ be two non-empty rational polyhedra where one is contained in the other, $\Delta^+ \supset \Delta^\circ$. Let $L_1, \dots, L_n \in \mathbb{L}_0(\mathbb{R}^e)$ be the rational semi-positive linear forms defining the faces of Δ° , $\Delta^\circ = \bigcap_{j=1}^n \Delta(L_j)$. Following [CP1] we set for every $j \in \{1, \dots, n\}$

$$l_j(\Delta^+) := \delta_{L_j}(\Delta^+).$$

By construction $0 \leq l_j(\Delta^+) \leq 1$ and if $l_j(\Delta^+) = 1$, then the difference of the Δ^+ to Δ° with respect to L_j is already zero. Hence the measure for the total difference is the non-negative rational number

$$\Lambda_{\Delta^\circ}(\Delta^+) := \sum_{j=1}^e \left(1 - l_j(\Delta^+)\right) \in \frac{1}{\beta! \alpha!} \mathbb{Z}_{\geq 0},$$

where β denotes the biggest denominator appearing in the coordinates of the (finitely many) vertices of Δ^+ and α is the biggest denominator appearing in the coefficients of L_1, \dots, L_n .

(3) If $\Delta^o = \Delta(J; u) \neq \emptyset$ and $\Delta^+ = \Delta(J; u; y)$, then we only write

$$l_j(J, u, y) := l_j(\Delta(J; u; y)) \quad \text{and} \quad \Lambda(J, u, y) := \Lambda_{\Delta(J; u)}(\Delta(J; u; y))$$

Remark 3.4. By Proposition 2.1 we can find in R a vertex-normalized (u) -standard basis $(f) = (f_1, \dots, f_m)$ and $\Delta(f; u; y) = \Delta(J; u; y)$. Note that $\Lambda(J, u, y) \in \frac{1}{\gamma! \alpha!} \mathbb{Z}_{\geq 0}$, where $\gamma = \max\{n_{(u)}(f_i) \mid 1 \leq i \leq m\}$. Since the numbers $n_{(u)}(f_i)$ are equal for any (u) -standard basis, we also have $\Lambda(J, u, z) \in \frac{1}{\gamma! \alpha!} \mathbb{Z}_{\geq 0}$ for any possible choice of (z) .

With this preparation we can now give the

Proof of Proposition 3.2. This is a slight generalization of the proof of THEOREM II.3 in [CP1].

Let $(f) = (f_1, \dots, f_m)$ be a vertex-normalized (u) -standard basis of J such that we have $\Delta(f; u; y) = \Delta(J; u; y)$. By Hironaka's theorem (Theorem 1.8) there exists a r.s.p. $(u_1, \dots, u_e, \hat{y}_1, \dots, \hat{y}_r)$ of \hat{R} such that (u, \hat{y}) is a r.s.p. for \hat{R} , (\hat{y}) yields $\text{Dir}(J')$, and $\Delta(J; u; \hat{y}) = \Delta(J; u)$. Let $\Lambda(J, u, y) := \sum_{j=1}^n (1 - l_j(J, u, y)) \geq 0$ be the measure introduced in Definition 3.3, where L_1, \dots, L_n are the semi-positive linear forms defining $\Delta(J; u)$.

As in [CP1] we follow Hironaka [H] (2.6) and consider the ideal of initial forms $In_{v_j}(J) = \langle in_{v_j}(g) \mid g \in J \rangle$ with respect to the monomial valuation $v_j := v_{L_j, u, y, f}$. The latter is defined by setting $I_\lambda := \langle u^A y^B \mid l_j(f, u, y) \cdot |B| + L_j(A) \geq \lambda \rangle \subset R$ for $\lambda \geq 0$ and $v_j(g) = \min\{\lambda \in \mathbb{Q} \mid g \in I_\lambda\}$ for $0 \neq g \in R$.

Let $a_{ij} \in \mathbb{Q}_{\geq 0}$ be the rational numbers defining L_j . Then we distinguish the system (u) in the following way: Let $\{u_i\}_{i \in I}$, $I \subset \{1, \dots, e\}$, be those elements of (u) for which $a_{ij} \neq 0$ and $\{u_{i'}\}_{i' \in I' = \{1, \dots, e\} \setminus I}$ the remaining elements.

Suppose $\Lambda(f, u, y) > 0$. Then $l_j(f, u, y) < 1$ for some (not necessarily all) $j \in \{1, \dots, n\}$. Fix j with this property and set

$$L := L_j, \quad \nu := v_j, \quad I := In_\nu(J) \quad \text{and} \quad l(f, u, y) := l_j(f, u, y).$$

We have two cases

- (1) $l(f, u, y) > 0$: Then $gr_\nu(R) = R/\langle y, \{u_i\}_{i \in I} \rangle[Y, \{U_i\}_{i \in I}]$; we set $t_{i'} := in_\nu(u_{i'}) \in gr_\nu(R)_0 = R/\langle y, \{u_i\}_{i \in I} \rangle$ for each $i' \in I'$ and $t_i := U_i \in gr_\nu(R)$ for all $i \in I$.
- (2) $l(f, u, y) = 0$: Then $gr_\nu(R) = R/\langle \{u_i\}_{i \in I} \rangle[[\{U_i\}_{i \in I}]]$; we set $t_{i'} := in_\nu(u_{i'}) \in gr_\nu(R)_0 = R/\langle \{u_i\}_{i \in I} \rangle$ for each $i' \in I'$, $Y := in_\nu(y) \in gr_\nu(R)_0$ and $t_i := U_i \in gr_\nu(R)$ for all $i \in I$.

In both cases we define

$$S := gr_\nu(R)_{\langle t_1, \dots, t_e, Y_1, \dots, Y_r \rangle}.$$

Then S is a regular local ring with r.s.p. $(t, Y) = (t_1, \dots, t_e, Y_1, \dots, Y_r)$ and residue field $S/N = K$, where $N := \langle t, Y \rangle$ denotes the maximal ideal of S .

The graded structure of $gr_\nu(R)$ induces a monomial valuation on S , again denoted by ν . Moreover, ν extends canonically to the N -adic completion \hat{S} of S .

We have inclusions $S \subseteq gr_\nu(\hat{R})_{\langle t, Y \rangle} \subseteq \hat{S}$ and an isomorphism

$$\hat{S} \cong \begin{cases} gr_\nu(\hat{R})_0[[Y, \{U_i\}_{i \in I}]] & \text{in case (1)} \\ gr_\nu(\hat{R})_0[[\{U_i\}_{i \in I}]] & \text{in case (2)} \end{cases}$$

By [EGA IV₂] (7.8.3)(ii), the localization of ring of finite type over a excellent ring is again excellent. Therefore S is excellent. By Proposition 2.1 we find a vertex-normalized (u) -standard basis $(f) = (f_1, \dots, f_m)$ in R and $\Delta(f; u; y) = \Delta(J; u; y)$. If we set $P_1 := in_\nu(f_1), \dots, P_m := in_\nu(f_m) \in S$, then $(P) = (P_1, \dots, P_m)$ is a vertex-normalized (u) -standard basis of I and thus $\Delta(P; t; Y) = \Delta(I; t; Y)$ (Lemma 2.2).

We set $\widehat{Y}_1 := in_\nu(\widehat{y}_1), \dots, \widehat{Y}_r := in_\nu(\widehat{y}_r) \in gr_\nu(\widehat{R})_{l_j(J, u, y)}$. Then (t, \widehat{Y}) is a r.s.p. for \widehat{S} and $\Delta(I; t; \widehat{Y}) = \emptyset$ which implies $\Delta(I; t) = \emptyset$.

Further $I \subset N$ and thus we can apply Hypothesis 3.1. Therefore there exist $Z_1, \dots, Z_r \in N$ such that (t, Z) is a r.s.p. for S and

$$\Delta(I; t; Z) = \emptyset.$$

As in the proof of THEOREM II.3 in [CP1]) we can now lift the elements $(Z) \subset S = gr_\nu(R)_{\langle t, Y \rangle}$ back to R and obtain (z_1, \dots, z_r) . Moreover, using the same arguments as they use we get the strict inclusion $\Delta(J; u; z) \subsetneq \Delta(J; u; y)$.

The change from (y) to (z) solves the whole face and all remaining points are closer to $\Delta(J; u)$. This means $0 \leq \Lambda(J, u, z) < \Lambda(J, u, y)$ and since $\Lambda(\cdot)$ takes only values in a discrete subset of $\mathbb{Q}_{\geq 0}$ this can only happen finitely many times.

Now we use Proposition 2.1 to get a vertex-normalized (u) -standard basis (h) of J with $\Delta(h; u; z) = \Delta(J; u; z)$. After that we repeat the arguments of this proof for $\Delta(J; u; z)$ instead of $\Delta(J; u; y)$.

By alternately applying this process we obtain in finitely many steps a desired vertex-normalized (u) -standard basis (g) of J and elements (z) in R extending (u) to a r.s.p. for R such that

$$\Delta(g; u; z) = \Delta(J; u).$$

□

Remark 3.5. In Example 1.15 we have seen that Hironaka's procedure is in general not finite. The problems may occur whenever $l(f, u, y) = 0$. If $l(f, u, y) > 0$ then Hironaka's procedure is finite and we obtain (z) by translating (y) by elements in R .

4. THE CASE OF AN EMPTY CHARACTERISTIC POLYHEDRON

Finally, we reduced the whole problem to the empty case. Thus it remains to show Hypothesis 3.1. Here we need to assume that (0.1) holds

Proposition 4.1. *Let R be a regular local Noetherian excellent ring, $J \subset R$ a non-zero ideal and $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$ a r.s.p. of R such that (y) determines the directrix of $J' = J \cdot R'$, where $R' = R/\langle u \rangle$.*

Suppose $\Delta(J; u) = \emptyset$ and assume that the radical of the ideal of the ridge of J' coincides with its ideal of the directrix,

$$\sqrt{IRid(J')} = IDir(J').$$

Then there exist a vertex-normalized (u) -standard-basis $(g) = (g_1, \dots, g_m)$ of J , elements $(z) = (z_1, \dots, z_r)$ in R such that (u, z) is a r.s.p. for R , (z) yields the directrix of J' , and

$$\Delta(g; u; z) = \Delta(J; u) = \emptyset.$$

Let us point out that (0.1) does in particular hold, when R contains a perfect field k .

Proof. By Proposition 2.1 there exist $(f) = (f_1, \dots, f_m)$ in R with $\Delta(f; u; y) = \Delta(J; u; y)$. Due to Hironaka (Theorem 1.8) there are (\widehat{y}) in \widehat{R} such that $\Delta(J; u; \widehat{y}) = \emptyset$. Since $\Delta(J; u; \widehat{y}) = \emptyset$ we get that $V(\widehat{y})$ is a permissible center for $V(\widehat{J})$, $\widehat{J} = J \cdot \widehat{R}$ (i.e. regular and $V(J)$ is normally flat along $V(\widehat{y})$ at every point of $V(\widehat{y})$, [CJS] Definition 2.1). Points where the Hilbert-Samuel function did not improve after blowing up in a permissible center (so called near points) are lying on the projective space associated to the reduced ridge of the tangent cone. Thus if we blow up with center $V(\widehat{y})$, there can not be any near points due to (0.1). This means we have blown up the whole Hilbert-Samuel stratum and its ideal in \widehat{R} coincides with $\langle \widehat{y} \rangle$.

By [CJS] Lemma 1.37(2) the Hilbert-Samuel stratum of $\widehat{J} = J \cdot \widehat{R}$ is solely determined by that of J . Hence there is an ideal $I \subset R$ such that $I\widehat{R} = \langle \widehat{y} \rangle$. Since R is excellent, R/I is regular and the height of I is r . Thus there exist regular elements $(z) = (z_1, \dots, z_r)$ in R such that $I = \langle z_1, \dots, z_r \rangle$. This implies $\langle z \rangle \widehat{R} = I\widehat{R} = \langle \widehat{y} \rangle$. By [CJS] $V(z)$ is permissible for $V(J)$ if

and only if $V(\hat{y})$ is permissible for $V(\hat{J})$. Therefore (u, z) is a r.s.p. for R , (z) determines the directrix of J' and we have the desired equality $\Delta(J; u; z) = \Delta(J; u; \hat{y}) = \Delta(J; u) = \emptyset$. \square

Remark 4.2 (*What can we say in the general case*). In general, (0.1) does only hold after a finite purely inseparable extension of the residue field K (see also the example below). If the differential operators map the ring R into itself, then one can deduce (0.1) by using this. But this is also not true in general (e.g. $R = \mathbb{Z}[X]_{(2, X)}$).

We have the following characterization of $V(\hat{y})$.

Claim: $V(\hat{y})$ is the unique permissible center for $V(J)$ of maximal dimension.

First of all, $V(\hat{y})$ is permissible. If there is a larger center containing $V(\hat{y})$, then we get a contradiction to the minimality of the generators of the directrix. If there is another component of the same dimension and transversal to $V(\hat{y})$, then we get again a contradiction to the property that the system (\hat{y}) yields the directrix. If there is another component of the same dimension and tangent to $V(\hat{y})$ and denote the corresponding ideal I' . Then we get that the associated polyhedron can not be empty and thus $g \notin I'^m$, i.e. $V(I')$ is not a permissible center.

Question: *Can we lift this center back to R ?* If there is an ideal $I \subset R$ such that $I \cdot \hat{R} = \langle \hat{y} \rangle$, then the excellence of R would imply again that I is regular and has height r . Hence there are elements $(z) = (z_1, \dots, z_r)$ in R such that $I = \langle z \rangle$ and all the desired properties hold.

Respectively, more general, not taking the previous characterization into account: *Is it possible to skip assumption (0.1), $\sqrt{IRid(J')} = IDir(J')$, in the statement of Proposition 4.1 and thus in Theorem A?*

In the situation over a field we can always attain condition (0.1) by passing to the algebraic closure of k . But by doing this the characteristic polyhedron may change drastically.

Example 4.3. Let k be field of characteristic $p \neq 2$ and set $q = p^e$ for some $e \in \mathbb{Z}_{\geq 0}$. Consider the variety given by

$$f = x^q + \lambda y^q + \lambda u_1^{aq} + \lambda^2 u_2^{bq},$$

where $\lambda, \lambda^2 \notin k^q$ are q -independent. If we consider the problem after the field extension $k' := k(t)/\langle t^q - \lambda \rangle$ over k , then $f = x^q + t^q y^q + t^q u_1^{aq} + t^{2q} u_2^{bq}$. Now condition (0.1) holds and the directrix is given by $z_0 := x + ty$. For $z := x + ty + tu_1^a + t^2 u_2^b$ we obtain $f = z^q$ and the characteristic polyhedron over k' is empty.

Another way of deducing (0.1) is by applying the derivative $\frac{\partial}{\partial \lambda}$. Then we stay in the local ring R and get $z_1 = x$ and $z_2 = y + u_1^b$. In R we can not solve the vertex corresponding to the monomial $\lambda^2 u_2^{bq}$.

Example 4.4. Let k be a field of characteristic $p > 0$. Consider the variety over k given by

$$f = y_1^p + \lambda y_2^p + u_1^{2p} + (\lambda + 1)u_2^{2p}$$

where $\lambda \notin k^p$. Again by applying the derivative $\frac{\partial}{\partial \lambda}$ we see that the desired elements are $z_1 = y_1 + u_1^2 + u_2^2$ and $z_2 = y_2 + u_2^2$. Using them we have $f = z_1^p + \lambda z_2^p$.

The following example (which is based on an example by Hironaka) illustrates that in general $V(\hat{y})$ is not a whole irreducible component of the Hilbert-Samuel locus. Moreover, it shows that taking the singular locus of the maximal Hilbert-Samuel locus does not characterize the ideal $\langle \hat{y} \rangle$.

Example 4.5. Consider the variety given by

$$f = x^2 + \lambda y^2 + \mu z^2 + \lambda \mu w^2 + y z u^{11},$$

over a field k , $\text{char}(k) = 2$ and $[k^2(\lambda, \mu) : k^2] = 4$. The order at the origin is $n = 2$, the ideal of the directrix is given by $\langle X, Y, Z, W \rangle$ and $f \in \langle x, y, z, w \rangle^2$. The derivatives are $\frac{\partial f}{\partial y} = z u^{11}$, $\frac{\partial f}{\partial z} = y u^{11}$, $\frac{\partial f}{\partial \lambda} = y^2 + \mu w^2$ and $\frac{\partial f}{\partial \mu} = z^2 + \lambda w^2$. Therefore the locus of maximal order (which coincides with the maximal Hilbert-Samuel locus because we are considering a hypersurface) is

$$V(x, y, z, w) \cup V(u, x^2 + \lambda \mu w^2, y^2 + \mu w^2, z^2 + \lambda w^2).$$

Note that the singular locus of this is the origin $V(x, y, z, w, u)$.

Example 4.6. Consider the variety given by

$$f = x^p + \lambda y^p + \lambda x^{p^2} + y^{p^2} + u_1^{p^3} + \lambda u_2^{p^3}$$

over a non-perfect field of characteristic $p > 0$, where $\lambda \in k \setminus k^p$. An idea would be to introduce weight in the coordinate y such that we artificially create condition (0.1). But if we do so, then we will never see that we have to solve y^{p^2} because it will be in the interior of the corresponding polyhedron.

It's not hard to see that the characteristic polyhedron is empty and the desired coordinates are $z_1 := x + y^p + u_1^{p^2}$ and $z_2 = y + x^p + u_2^{p^2}$.

5. ON THE CHARACTERISTIC POLYHEDRA OF IDEALISTIC EXPONENTS

In [Sc2] the second author introduced a characteristic polyhedron for idealistic exponents which is related to Hironaka's characteristic polyhedra. As a corollary of the previous result we obtain that under the same assumptions (R excellent and (0.1)) the characteristic polyhedra of an idealistic exponent can also be attained in the regular local ring R without passing to the completion.

For this we do not need to recall the whole theory of idealistic exponents. It suffices to consider pairs (J, b) , where $J \subset R$ is an ideal and $b \in \mathbb{Q}_+$ a positive rational number.

Definition 5.1. Let R be a regular local Noetherian excellent ring, $J \subset R$ a non-zero ideal, and $b \in \mathbb{Q}_+$ a rational number which is smaller or equal than the order of J in the maximal ideal. Further, let $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$ be a r.s.p. of R such that (y) determines the directrix associated to $\langle in(g, b) = g \bmod M^{b+1} \mid g \in J \rangle$. Let $(f_1, \dots, f_m) \subset R$ be any set of generators of J and expand $f_i = \sum C_{A,B,i} u^A y^B$. We define the polyhedron $\Delta((J, b); u; y)$ associated to the pair (J, b) and (u, y) as the smallest F -subset containing the points of the set

$$\left\{ \frac{A}{b - |B|} \mid C_{A,B,i} \neq 0 \wedge |B| < b \right\}$$

The *characteristic polyhedron of the pair (J, b) and (u)* is then defined by

$$\Delta((J, b); u) = \bigcap_{(y)} \Delta((J, b); u; y),$$

where the intersection ranges over those (y) so that (u, y) is a r.s.p. of R and (y) determines the directrix above.

Note that in the definition of $\Delta((J, b); u; y)$ we have $n_i - |B|$ in the denominator. Hence these two polyhedra can differ (see [Sc2] Example 5.7). But both polyhedra are certain projections of the Newton polyhedron which is the polyhedron defined by the points $(A, B) \in \mathbb{Z}^{e+r}$. One shows easily that $\Delta((J, b); u; y)$ is independent of the choice of the generators (see loc. cit. Corollary 4.4). Therefore we only have to convince ourself that the process of solving vertices is also finite for $\Delta((J, b); u)$.

Proposition 5.2. *Let the situation be as in the previous definition. Suppose (0.1) holds.*

Then there are elements $(z) = (z_1, \dots, z_r)$ in R such that (u, z) is a r.s.p. of R , (z) yields the directrix of (J, b) and

$$\Delta((J, b); u; z) = \Delta((J, b); u).$$

Proof. Up to a small modification because of b the proof is word by word the same as for Proposition 4.1 and Proposition 3.2. \square

In [Sc1] Construction 2.3.2 there is an alternative definition of $\Delta((J, b); u)$ given in which one deduces this characteristic polyhedra from Hironaka's. This is another way to see that the previous proposition is a corollary of Theorem A.

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